Chapter 4. Continuous Random Variables

Intro: §4.1 Probability Density Functions (pdf)

§4.2 Cumulative Distribution Functions (cdf)

FACT: If X is a continuous R.V. then
$$P(X=x)=0$$

Need a replacement for probability mass funct...

But definition of cdf is still fine:

Def: Continuous R.V. X has cumulative dist. fun
$$F(x)=P(X\le x)$$

Note: $P(X=x)=0$, so now
$$P(X\le x)=P(X<x) \quad !!$$

Recall: We can compute pmf from cdf for discrete R.V. by computing change in cdf:
$$f(x_n)=F(x_n)-F(x_{n-1})$$
$$P(X=x_n)=P(X\le x_n)-P(X\le x_{n-1})$$

For continuous R.V. change should be computed with derivative!

Def: Continuous R.V. has prob. density fun.
$$f(x)=\frac{d}{dx}F(x)$$

Note: $F(x)$ is an antideriv. of $f(x)$.

Since $F(x)=0$ for $x\ll 0$ we get:
$$P(X\le b)=F(b)=\int_{-\infty}^{b}f(x)\,dx$$

— and —
$$P(a\le X\le b)=F(b)-F(a)=\int_a^b f(x)\,dx$$

WARNING: Unlike for discrete R.V., the pdf
$$f(x)\ne P(X=x)$$
Probability that $X=x$

Instead it is more like
$$f(x)\approx P(x-\varepsilon\le X\le x+\varepsilon)\cdot\frac{1}{2\varepsilon}$$
"Probability that $X\approx x$"

Properties of pdf & cdf

pdf:

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

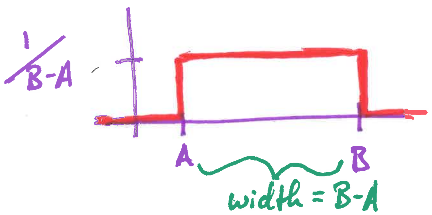
cdf:

- $F(x) \approx 0$ for $x \ll 0$
- $F(x) \approx 1$ for $x \gg 0$
- $F(x)$ is non-decreasing (If $a < b$ then $F(a) \leq F(b)$)

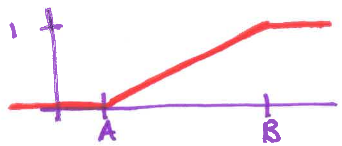
The pdf $f(x)$ is (usually) much easier to define and understand than the cdf $F(x)$
— same as for discrete R.V.

Example "Uniform Distribution on interval $[A, B]$"

has pdf
$$f(x) = \begin{cases} \frac{1}{B-A} & \text{if } A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

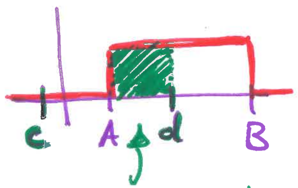


$f(x)$



$F(x)$

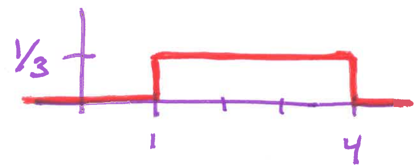
In this case
$$P(c \leq X \leq d) = \int_c^d f(x)\,dx$$
$$= \frac{\text{amount of } [c,d] \text{ inside } [A,B]}{\text{length of } [A,B]}$$



$P(c \leq X \leq d)$

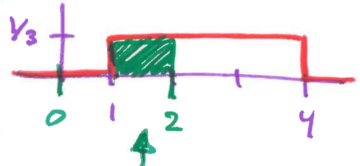
Numeric Example. $X \sim \text{Uniform}([1, 4])$

Then pdf is



$P(0 \leq X \leq 2)$ is

Area (



)

$\frac{1}{4-1} = 1/3$

Expected Value & Variance are defined similarly:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

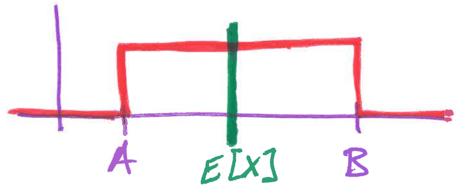
$$E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

$$\mathrm{Var}[X] = E[(X-\mu)^2]$$

. $E[X]$ is the "<u>center of mass</u>" of $f(x)$

For example, if $X \sim \text{Uniform}([A, B])$

then $E[X] = \frac{A+B}{2}$



Note: $E[X]$ and $\text{Var}[X]$ are usually pretty unpleasant to compute by hand....